Infinite Matrices, Ritz Theorem and Bound State Problems

A. K. Mitra

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409

(Z. Naturforsch. **30 a**, 256–261 [1975]; received November 15, 1974)

The straight forward application of the Ritz variational technique has been shown to be a very convenient method for obtaining numerically the first few discrete eigenvalues of the Schroedinger operator with certain special types of potentials. This method solves essentially the (finite) matrix eigenvalue problem obtained by truncating the infinite matrix representing the Schroedinger operator with respect to the Coulomb wave functions. The Ritz theorem justifies the validity of this truncation procedure.

I. Introduction

The initial motivation behind the present work was to obtain numerically the first few energy levels of two electrostatically bound (hydrogen-like atoms) spherical clouds of uniform charge. In the conventional hydrogen-like model, the orbiting particle (the lighter one) is usually considered as a point particle, whereas the heavier one is assumed to have a certain geometric shape or to extend to all space with a given type of charge distribution [1]. It may therefore seem useful, at least from a theoretical view point, to extend the considerations to the case in which the orbiting particle also has a finite dimension, possibly a sphere with a uniform charge distribution. On the other hand, this may also prove useful in the quantum theory at small distances-where the simple notion of a "point-particle" may not be valid anymore. For example, the muon being about 210 – times heavier than the electron, its Bohr orbit may lie well within the nucleus, as a result of which the nucleus can not be considered as a point particle anymore [2]. Whereas a considerable amount of work has been done with regard to muonic atoms, the hitherto considered models have regarded the muon as a point particle. The two-sphere bound state problem may be applicable as well to the muonic atom, where the muon can be considered also as a uniformly charged spherical cloud of radius R_μ ($\ll R_N$, the nuclear radius). From the comparison of the corresponding theoretical transition energies (obtained from constructive models), with the experiment, one might obtain an initial lower bound or an estimate for the muon radius R_μ.

This is the main aim of this work. However the present paper does not solve this problem to the full extend- and therefore it should be considered as an initial exposition of the numerical method to be employed for this purpose. The extension to the two sphere problem will be then evident and corresponding results will be presented in a short note as a continuation of the present work.

According to the proposed model we want to find the discrete eigenvalues of the Schroedinger operators defined on a dense linear manifold of a Hilbert space. Among several methods of solving eigenvalue problems in Hilbert space, the variational method, in one or another form, is known to be one of the most effective techniques [3, 4] for obtaining approximate eigenvalues of semi-bounded operators. On the other hand, some other approximation method may, in some way, be related to the variational method [5]. We have chosen here one of the simplest and classic forms of the variational method; i. e., the Ritz method as exposed in Ref. [3] (pp. 183). Whereas the Ritz method usually finds upper bounds for the eigenvalues, the Weinstein method, in its original form [4], finds lower bounds. The latter technique has been successfully used by Bazley and Fox in connection with the helium atom and the anharmonic oscillator (this has been discussed in detail in Reference [4]).

The main point of this paper is to obtain the infinite symmetric matric representation of the Schroedinger operator with respect to the Coulomb wave functions. The eigenvalues are then obtained from the corresponding truncated $N \times N$ forms of the infinite matrix, where N is chosen large enough to yield the desired degree of accuracy. The validity of such a truncation procedure is given by the Ritz theorem (the mathematical preliminaries will be discussed in Section 2). Among several standard algorithms for finding approximate eigenvalues of a real symmetric $N \times N$ matrix, the Givens-Householder method [6, 7] appears to be the simplest one,

and has been used in this paper. The case $N=50$ has sufficed to yield accuracy to at least five decimal places.

It is to be noted that the method (the Ritz method in combination with the Givens-Householder method) is not basically new. It is just that this method, in its present form, has not been in frequent use in quantum mechanics. Also this method has been very successfully applied to the λx^{2m} anharmonic oscillator [8]. In this paper the method has been illustrated by choosing the potential V_p for the point muon (already discussed in Ref. [2]):

$$V_p(r) = -\frac{Ze^2}{R}\left(\frac{3}{2}-\frac{1}{2}\bar{r}^2\right) \quad (\bar{r}\leqq 1)$$
$$= Ze^2/R\,\bar{r} \quad (\bar{r}\geqq 1) \qquad (1.1)$$

where $\bar{r}=r/R$, R being the nuclear radius. The reasons for choosing such a potential are the following: (a) to apply a perturbation method to the two-sphere problem, where V_p will be considered as the unperturbed potential, the corresponding unperturbed eigenfunctions being obtained as a truncated Fourier series in the Coulomb wave functions; (b) a straight forward extension of the present variational method to the two-sphere problem, where the corresponding potential $V_F(r)$ has a relatively complex form due to the possible overlap of the two-spheres (regarded as charged clouds). However V_F depends essentially on a single parameter $s=R_\mu/R_N$. The main problem therefore will be to compute the matrix elements of the potential V_F in the chosen basis.

The numerical results will be presented in Section 3. It is expected that the method will be applicable to the more general and recently used potentials in the Schroedinger equation (for example, the case where the nucleus has a Fermi-type charge distribution). Also little is known about the variational approach of the above kind for the Dirac operator [9] – and the attempt as such seems worthwhile.

II. The Mathematical Preliminaries

A) The Representation of the Schroedinger Operator

The material presented in this section is the well-known technique of finding the representation of the Schroedinger operator in a chosen basis. The main purpose of this section has been therefore to obtain the form of the matrix elements of the Schroedinger operator suitable for numerical calculations.

It is to be noted that the present discussion applies only to the discrete eigenvalues of the Schroedinger operator. That is we shall be considering in a way a restriction S of the operator with its domain as the dense linear manifold M spanned by the orthonormal set of the hydrogen wave functions $R\,n\,l(r)\,y_{lm}(\theta,\Phi)$ [see Appendix A], which belong to $L_2(R^3)$. Then we solve the eigenvalue equation $Su=\lambda u$, with $u\,\varepsilon\,M$. λ belongs to the discrete eigenvalues with the possible bound state given by u as a linear combination of the hydrogen wave functions. The existence of the solution of such an eigenvalue equation restricts our discussion to discrete eigenvalues only and therefore the continuum does not enter into our discussion.

We introduce the following dimensionless quantities $\alpha, \bar{V}(r), \lambda, \varrho$:

$$\alpha = ZR/a\,,\ V(r) = -\varepsilon_0\,\bar{V}(r)\,,\ E = -\varepsilon_0\,\lambda\,,$$
$$\varrho = 2\,\alpha\bar{r}\,, \quad (\bar{r}=r/R)\,. \qquad (2.1)$$

Here $a=h^2/\mu e^2 \approx 255.92\,\mathrm{F}$ is the muon Bohr radius, r being the muon mass, and $\varepsilon_0 = 2Z^2e^2/a$ is an energy unit depending on Z. E is the energy eigenvalue of the Schroedinger operator. Then the Schroedinger equation in a potential given by (1.1) can be put in the following form [2]

$$Su = Hu + \varrho^2\,\bar{V}(\varrho)\,u = \lambda\,\varrho^2\,u \qquad (2.2)$$

where

$$H = \frac{\partial}{\partial\varrho}\left(\varrho^2\,\frac{\partial}{\partial\varrho}\right) + H_1\,;$$
$$H_1 = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\Phi^2} \qquad (2.3)$$
$$\bar{V}(r) = \frac{1}{2\,\alpha}\left(\frac{3}{2}-\frac{1}{2}\bar{r}^2\right), \quad (\bar{r}\leqq 1)$$
$$= 1/2\,\alpha\bar{r} \quad (\bar{r}\geqq 1)\,.$$

Here we consider the usual boundary conditions as in the case of a Coulomb potential. It is to be noted that for the bound state problem ($E<0$), λ defined above will be positive, and therefore the highest eigenvalue λ of (2.3) will give the lowest energy level and so on.

As the domain of H we consider the dense linear manifold M spanned by the complete orthonormal set of the Coulomb wave functions [Ref. [10], see also

Appendix (A)]

$$u_{nlm} = R_{nl}(r)\, Y_{lm}(\theta, \Phi) \qquad (2.4)$$
$$(n = 1, 2, \ldots; \quad l = 0, 1, 2, \ldots; \quad -l \leqq m \leqq l)$$

where, according to the definition of the inner product.

$$\langle f, g \rangle = \int f g \, \mathrm{d}\boldsymbol{r} \qquad (2.5\,\mathrm{a})$$

we have

$$\langle u_{nlm}\, u_{n'l'm'} \rangle = \delta_{nn'}\, \delta_{ll'}\, \delta_{mm'}\,. \qquad (2.5\,\mathrm{b})$$

In order to solve (2.1) we consider the matrix representation $\widetilde{S}$ of S in the basis $\{u_{nlm}\}$. Using the usual differential operator for the Coulomb potential, an easy calculation shows that

$$H\, u_{nlm} = \left[\frac{\varrho^2}{4(n+l)^2} - \varrho\right] u_{nlm} \qquad (2.6)$$

[as for the Coulomb potential the energy eigenvalues are given by [10]

$$E_{nl} = -\left[\frac{\mu\, Z^2\, e^4}{2\,\hbar^2 (n+l)^2} = -\frac{\varepsilon_0}{4(n+l)^2}\right].$$

Thus, using (2.2) through (2.6) the eigenvalue Eq. (2.2) can be put in the following algebraic form involving the infinite matrix $\widetilde{S}$ as the representation of S:

$$\widetilde{S}\, X = \lambda\, X \qquad (2.7)$$

where the elements of the matrix $\widetilde{S}$ are given by

$$\langle n\, l\, m \,|\, \widetilde{S} \,|\, n'\, l'\, m' \rangle = \left(\frac{\delta_{nn'}}{4(n+l)^2} - G_{nn'}\right) \delta_{ll'}\, \delta_{mm'} \qquad (2.8)$$

with

$$G_{nn'} = \frac{R^3}{2\,\alpha} \int_0^1 \left(\bar{r} - \frac{3}{2}\,\bar{r}^2 + \frac{1}{2}\,\bar{r}^4\right) R_{nl}(r)\, R_{n'l}(r)\, \mathrm{d}\bar{r}. \qquad (2.9)$$

Also, X is the eigenvector of $\widetilde{S}$ and contains the Fourier coefficients A_{nlm}'s in the eigenfunction expansion of u:

$$u = \sum A_{nlm}\, u_{nlm}\,. \qquad (2.10)$$

A method of evaluation of the integral (2.9) is given in the Appendix A. The integral in (2.9) is from 0 to 1 (instead from 0 to ∞), because $G_{nn'}$ is the matrix element of the function $\varrho - \varrho^2 \bar{V}$ which vanishes identically for $\bar{r} \geqq 1$.

A wel-known feature emerges from the Eq (2.8), namely, the representation of S in the chosen basis decomposes into a block diagonal form, whereas the manifold M decomposes into the direct sum of orthogonal subspaces each of which is invariant under S. This block diagonalization can be realized in the following way:

We reorder the basic vectors u_{nlm} in such a way that to each l there correspond $2l+1$ infinite dimensional subspaces corresponding to the $2l+1$ values of m, each of which is spanned by the set $\{u_{nlm}\}_{n=1}^{\infty}$ for fixed l, m. Then the matrix $\widetilde{S}$ decomposes, according to (2.8) as:

$$\widetilde{S} = \left[\begin{array}{c|c|c|c|c} l=0 & & & & \\ \hline & m=-1 & & & \\ \hline & & m=0 & & \\ \hline & l=1 & & m=1 & \\ \hline & & & & \ddots \end{array}\right] \qquad (2.11)$$

Here the $2l+1$ submatrices for different m but fixed l are the same due to the well-known m-degeneracy in a central potential. The eigenvalues of the submatrix with $l=0$ correspond to the s-states, and those of the submatrix with $l=1$ to the p-states and so on. For example the highest eigenvalue (λ_1) corresponding to $l=0$ gives the energy of the 1s-state, whereas the second highest (λ_2) for $l=1$ will give the energy of the 2p-state. It remains therefore to find numerically the first few eigenvalues of such infinite submatrices. An attractive method is naturally to truncate the infinite matrices to the corresponding finite $N \times N$ forms and then to solve the eigenvalue problem for such finite matrices, for which N has to be large enough so that

$$\lambda_k^{(N+1)} - \lambda_k^{(N)} < \varepsilon$$

where $\lambda_k^{(N)}$ is the k-th eigenvalue of the $N \times N$ form and ε is an approximation parameter.

The validity of such a truncation procedure is given by the following Ritz Theorem.

B) The Ritz Theorem

Among several available monographs on the Ritz variational method we follow here the excellent exposition due to Michlin and Gould [3, 4]. Let A, with domain D_A, be a symmetric operator with descrete eigenvalues and bounded from above i. e.:

$$\langle A\, u, u \rangle < \beta, \text{ with } \langle u, u \rangle = 1\,. \qquad (2.12)$$

Let $\{u_n\}$ be an orthonormal subset of functions in D_A, complete with respect to the energy norm defined by

$$||u||_E^2 = \langle \bar{A}u, u \rangle \qquad (2.13)$$

where $\bar{A} = \beta I - A$, I being the identity operator [clearly $\bar{A}$ is positive definite i.e. $\langle \bar{A}u, u \rangle > 0$].

If (A_{mn}) be the matrix representation of A with respect to the basis $\{u_n\}$, then the Ritz theorem asserts the following:

Let $\lambda_1^{(N)} > \lambda_2^{(N)} > \ldots > \lambda_k^{(N)} > \ldots$ be the ordered eigenvalues of the truncated $N \times N$ form $A^{(N)}$ of (A_{nm}).

Then $\lambda_k^{(N)} \leqq \lambda_k^{(N+1)}$ and $\lambda_k^{(N)} \to \lambda_k$ as $N \to \infty$, where λ_k is the exact k-th eigenvalue of the operator A.

Therefore the theoretical process of truncation in our case for the approximate numerical estimation of the eigenvalues of $\tilde{S}$ will be perfectly justified if we can show that S satisfies the conditions of the Ritz theorem. This is proved in Appendix B.

III. The Numerical Results

In this section we find numerically the energy eigenvalues of the 1s and 2p states (i.e. the first and the second eigenvalues of the block corresponding to $l=0$ and $l=1$ respectively). We do this by selecting the corresponding truncated forms $N \times N$, for different values of N, until the agreement to the required degree of accuracy has been obtained.

As for the numerical method for finding the eigenvalues of finite matrices, there exist, to this date, several methods [6]. For a real symmetric matrix, the Givens-Householder method appears to be a very effective one and so this has been used in this paper. The computer algorithm has been similar to *the one* used by Ortega [7]. [Another iteration method has also been used: This method finds all the eigenvalues and the eigenvectors of the $(N+1) \times (N+1)$ form from the knowledge of the corresponding quantities for the $N \times N$ form. This is done again by reducing the $(N+1) \times (N+1)$ form to a border diagonal form [6]. This method is, however, significantly slower than the other known methods and it does not seem useful to go into further details. The main point to be noted is that the numerical results obtained by both of these methods agree up to the degree of accuracy considered here.] The corresponding upper bound (to be used in the Givens-Householder method) turns out to be 0.25 in this case (see Appendix B). Also the matrix elements depend only on the single parameter α (see Appendix A).

In fact, $N=50$ has been sufficient to yield the first three eigenvalues for $l=0, 1$, accurate at least to five decimal places. Actually, for higher l or smaller α the convergence has been significantly faster – even $N=10$ has been enough. For this reason we have considered, in the examples below, only certain nuclei with medium or high Z. Also the first few eigenvalues have been calculated for higher values of l, but have not been reproduced here. In the Tables I and II, the results for 1s and 2p states are given for $R = 1.3\,A^{1/3}F$ and $R = 1.5\,A^{1/3}F$ respectively. The corresponding $2p-1s$ transition energies are obtained by multiplying the difference $\lambda_1 - \lambda_2$ of the eigenvalues by the corresponding energy unit $\varepsilon_0 = 2Z^2e^2/a$. Taking $e = 4.80286 \times 10^{-10}$ esu, $a = 255.92\,F$ we have $\varepsilon_0 = 11.252443 \times Z_2$ keV. These transition energies are given in Table III and are compared with the results of Cooper and Henley and with the experiment [11].

Table I. The values of $\alpha = ZR/a$ and the eigenvalues λ_1 and λ_2, for the 1s and 2p states with $R = 1.3 \times A^{1/3}$.

Nuclei	α	λ_1	λ_2
$^{40}_{19}$K	.330075	.233600	.062500
$^{63}_{29}$Cu	.586161	.208868	.027765
$^{90}_{40}$Zr	.910570	.176407	.027714
$^{121}_{51}$Sb	1.281360	.147204	.027575
$^{140}_{58}$Ce	1.529830	.133557	.027420
$^{202}_{80}$Hg	2.384402	.108960	.026460
$^{206}_{82}$Pb	2.460039	.107671	.026346
$^{208}_{82}$Pb	2.467975	.107540	.026334

Table II. The values of $\alpha = ZR/a$ and the eigenvalues λ_1 and λ_2 for the 1s and 2p states with $R = 1.5 \times A^{1/3}$.

Nuclei	α	λ_1	λ_2
$^{40}_{19}$K	.380856	.229125	.062492
$^{63}_{29}$Cu	.676340	.199558	.027756
$^{90}_{40}$Zr	1.050658	.164109	.027674
$^{121}_{51}$Sb	1.478492	.136029	.077457
$^{140}_{58}$Ce	1.765188	.124132	.027221
$^{202}_{80}$Hg	2.751234	.103339	.025870
$^{206}_{82}$Pb	2.838507	.102193	.025718
$^{208}_{82}$Pb	2.847663	.102077	.025702

Table III. 2p−1s Transition Energies: $\varepsilon_1=(\lambda_1-\lambda_2)\,\varepsilon_0$, as calculated from Tables I and II, ε_{CH}, as given by Cooper and Henley and ε_E, the experimental values (the later two given as $2p\,\frac{1}{2}-1s\,\frac{1}{2}$ transition energies. For ε_{CH}, R chosen was $1.3\times A^{1/3}$).

Nuclei	ε_1 (keV) ($R=1.3\,A^{1/3}$)	ε_1 (keV) ($R=1.5\,A^{1/3}$)	ε_{CH} (MeV) ($R=1.3\,A^{1/3}$)	ε_E (keV)
$^{40}_{19}$K	719.40	676.88		712.64 ± 0.23
$^{63}_{29}$Cu	1713.83	1625.81	1.51	1511.4 ± 1.0
$^{90}_{40}$Zr	2677.06	2456.36		2535.9 ± 0.4
$^{121}_{51}$Sb	3501.25	3177.64	3.37	3543.3 ± 2.0
$^{140}_{58}$Ce	4017.63	3668.39		4160.3 ± 5.0
$^{202}_{80}$Hg	5941.29	5578.98		5645.1 ±10.0
$^{206}_{82}$Pb	6153.17	5786.21	5.30	5786.9 ± 5.0
$^{208}_{82}$Pb	6144.16	5778.64		5783.7 ± 5.0

The calculated transition energies do not agree very well with the experimental values for all the nuclei given in the table. The calculated values depend appreciably on the parameter α or equivalently on the radius R. The correct experimental values for each individual nucleus might indicate the actual discrepancies given by this model. On the other hand we have neglected the effect of the spin-orbit coupling, which will contribute to a certain extent for the heavier nuclei. It may also be noted that the isotope shifts can be similarly be calculated. Finally, it is believed that the muonic atoms are better described by the approximate Dirac equation with a Fermi charge distribution for the nucleus. The author hopes to discuss this more general problem based on the method given in this article.

Appendix A

Here we give the numerical method of evaluating the matrix elements G_{nm} defined by Equation (2.9). Here, following Ref. [10], we have

$$R_{nl}=\sqrt{\frac{Z(n-1)!}{(n+l)^2\,(n+2\,l)!\,a}}\;\varrho^{l+1}\,e^{-\varrho/2}\,L_{n+2}^{2l+1}(\varrho) \tag{A 1}$$

where

$$\varrho=(2\,\alpha/n+l)\bar{r}\,.$$

The Laguerre polynomial $L_{n+2l}^{(2l+1)}(\varrho)$ is defined by

$$L_{n+2l}^{2l+1}(\varrho)=\sum_{k=0}^{n-1}a_K\,\bar{r}^K$$

with

$$a_K=\frac{\sqrt{(n-1)!\,(n+2\,l)!}\;\;(2\,\alpha)^K}{(n-1-K)!\,(2\,l+1+K)!\,K!\,(n+l)^{K+2}}\,. \tag{A 2}$$

In order to evaluate the integral for G_{mn} we first multiply the corresponding Laguerre polynomials and then integrate each term in the products $R_{nl}R_{ml}$ which are of the form $\bar{e}^{\theta\bar{r}}\,\bar{r}^K$, where $\theta=[1/(n+l)+1/(m+l)]\,\alpha$. The rule of multiplication of two finite series of the form

$$s_1=\sum_{K=0}^{p(n)}a_K\,x^K \quad\text{and}\quad s_2=\sum_{K=0}^{q(m)}b_K\,x^K$$

is given by

$$s=s_1\,s_2=\sum_{L=0}^{p(n)+p(m)}C_L\,x^L$$

where the coefficients C_L are given by [assuming $p(m)\leqq p(n)$]

$$C_L=\sum_{K_1}^{K_2}a_K\,b_{L-K} \tag{A 3}$$

where

$K_1=0,\ K_2=L$ if $L\leqq p(m)$,

$K_1=L-p(m),\ K_2=L$ if $p(m)\leqq L\leqq p(n)$,

and

$K_1=L-p(m),\ K_2=p(n)$ if $p(n)\leqq L\leqq p(n)+p(m)$.

In the above case $p(n)=n-1$ and $p(m)=m-1$, whereas the b_K's are obtained from the a_K's in (A3) by replacing n by m. Then G_{mn} is given by

$$G_{mn}=\tfrac{1}{2}\sum_{L=0}^{m+n-1}(-2\,\alpha)^{J+2}\,C_L[X(J+1)-\tfrac{3}{2}\,X(J+2)+\tfrac{1}{2}\,X(J+4)] \tag{A 4}$$

$$(J=L+2\,l)$$

where

$$X(L)=\int_0^1 e^{-\theta\bar{r}}\,\bar{r}^L\,\mathrm{d}\bar{r}=\frac{L!}{\theta^{L+1}}\left[1-\bar{e}^{\theta}\sum_{K=0}^{L}\frac{\theta^K}{K!}\right]$$

For large values of m and n the series in (A 4) can be truncated long before the actual upper limit is attained, since the C_L's decrease quite rapidly and the functions $X(L)$ are smaller than 1. This way the computation is speeded significantly and the overflow and underflow can be avoided which may occur due to large values of the factorials. Also, G_{mn} depends only on a single parameter, namely α, and the computation is much faster for smaller α and larger l.

Appendix B

We first show that the Schroedinger operator S defined by (2.2) is bounded from above. Following Sec. III [Compare Eqs. (2.6) and (2.9)], we write

$$S = H + \varrho - G = H_0 - G \qquad \text{(A 1)}$$

where $H_0 = H + \varrho$ is the corresponding operator for the Coulomb potential, and

$$G(\bar{r}) = \frac{R^3}{2\alpha}\left(\bar{r} - \frac{3}{2}\bar{r}^2 + \frac{1}{2}\bar{r}^4\right) \quad (\bar{r} \leqq 1)$$
$$= 0 \quad (\bar{r} \geqq 1)\,. \qquad \text{(A 2)}$$

It is easy to see that $\langle u, G u\rangle \geqq 0$, since $G(\bar{r}) \geqq 0$. Thus $\langle u, S u\rangle \leqq \langle u, H_0 u\rangle$. But by (2.9) and definition of H_0, it is clear that $\langle u, H_0 u\rangle < \beta = 1/4$.

Further, let $G_{\max}$ be the maximum value of $G(\bar{r})$. Then by (2.9) we have,

$$\langle v, H_0 v\rangle = \frac{R^3}{2\alpha} G_{\max} \| u \|^2 = g \| u \|^2\,, \qquad \text{(A 3)}$$

and hence,

$$\langle u, S u\rangle \geqq \langle u, H_0 u\rangle - g \| u \|^2\,. \qquad \text{(A 4)}$$

Let $A = \beta I - S$, $v = \sum_{N+1}^{\infty} a_{(n)}\, u_{(n)}$, where

$$a_{(n)} = \langle u_{(n)}, u\rangle \quad \text{and} \quad (n) = n\,l\,m\,.$$

Then by (A 4), we have

$$0 \leqq \langle v, A v\rangle \leqq (\beta + g) \| v \|^2 - \langle v, H_0 v\rangle\,. \qquad \text{(A 5)}$$

But by (2.9), we have

$$\langle v, H_0 v\rangle = \sum_{N+1}^{\infty} \frac{a_n{}^2}{4(n+l)^2} \to 0 \text{ as } N \to \infty$$

since $\{u_{nlm}\}$ is already known to be complete with respect to the standard norm defined by (2.5). For the same reason, the first term in (4.5) also vanishes as $N \to \infty$. This proves the completeness.

[1] Y. N. Kim, Mesic Atoms and Nuclear Structure, North-Holland Publishing Co., Amsterdam 1971.
[2] L. N. Cooper, E. N. Henley, Phy. Rev. **92**, 801 [1953].
[3] S. G. Michlin, Variationsmethoden der Mathematischen Physik, Akademie Verlag, Berlin 1962.
[4] S. H. Gould, Variational Method for Eigenvalue Problems, University of Toronto Press, 1964.
[5] George A. Baker, Jr., and John L. Gammel, The Pade' Approximant in Theoretical Physics, Academic Press, New York 1970.
[6] J. H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford 1965.
[7] A. Ralston and H. S. Wilf, ed., Mathematical Methods for Digital Computers, Vol. II, John Wiley and Sons, Inc., New York 1967.
[8] A. K. Mitra, The Eigenvalues of λx^{2m} Anharmonic Oscillator (unpublished).
[9] T. Kato, Perturbation Theory of Linear Operators, Springer-Verlag, New York 1966.
[10] Amos De-Shalit and Igal Talmi, Nuclear Shell Theory, Academic Press, New York 1963.
[11] M. Baranger and E. Vogt, Advances in Nuclear Physics, Plenum Press, New York 1968.